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RADIAL OSCILLATIONS OF NONHOMOGENEOUS, THICK-WALLED CYLINDRICAL AND SPHERICAL SHELLS SUBJECTED TO FINITE DEFORMATIONS

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Abstract—The infinitesimal breathing motions of long cylindrical tubes and hollow spherical shells of arbitrary wall thickness subjected to a finite deformation field caused by uniform internal and/or external pressures are investigated. A neo-Hookean material with a material constant varying continuously along the radial direction is used. The shell is first subjected to finite static deformations and is then exposed to a secondary dynamic displacement field. Based on the theory of small deformations superposed on large deformations, closed form expressions are obtained for the frequency of small oscillations about the highly prestressed state. Frequency versus initial deformation parameter curves are given for several nohomogeneity functions and for various wall thicknesses.

INTRODUCTION

The static and dynamic behaviors of thick and thick-walled elastic bodies undergoing large deformations have been investigated extensively by several authors. The rigorous finite elasticity theory in conjunction with the theory of small deformations superposed on large deformations [1] has been used to analyze the stability and the vibrational characteristics of such bodies [2–7]. The early works of Knowles [8, 9] constituted the basis for later papers [10, 11] on finite amplitude oscillations. In most cases, the material of the body is assumed to be isotropic, elastic, incompressible and homogeneous. To a certain extent, Nowinski and Shahinpoor [12] considered the effect of nonhomogeneity in their study concerned with the breathing motions of hollow spherical shells of arbitrary wall thickness subjected to finite deformations caused by a uniform internal pressure. In [12], a continuous nonhomogeneity is introduced; the material of the shell is of neo-Hookean type with the material constant varying as a function of the radial distance. However, the results obtained in [12] indicate, incorrectly, an erratic behavior of the shells.

The questionable results of [12] have led the present authors to re-examine the problem and also extend the investigation to circular cylindrical shells. In the present work, the effect of initial internal and/or external pressures on the free breathing motions of hollow cylindrical and spherical shells is studied. The material of the shells is assumed to be of neo-Hookean type, with the material constant varying as a function of the radial coordinate. For cylindrical shells, both the initial and the secondary deformation fields are assumed to have axial symmetry. For spherical shells, both fields are taken to be spherically symmetric. The theory of small displacements superposed on large elastic deformations [1] is used in the formulation of the problems. More specifically, the deformation and the stress fields of the elastic body under the external forces are determined by a semi-inverse procedure and then an infinitesimal dynamic displacement field is superposed on the previously determined equilibrium state. The field equations governing the small oscillations superposed on large elastic deformations have closed form solutions for both cylindrical and spherical shells. The characteristic equations of both problems, which contain the frequencies of small oscillations as the unknown, are obtained from their respective boundary conditions.

FORMULATION OF THE PROBLEMS

1. *Cylindrical shells*

Consider a long, circular cylindrical shell of arbitrary wall thickness with an inner radius A_1 and an outer radius A_2 in the undeformed state. Assume that the material of the shell is isotropic, perfectly elastic and incompressible with a strain energy density function given by $W = C_1(I-3)$ where I is the first strain invariant and C_1 is the only material constant. If the material is nonhomogeneous with respect to the radial coordinate then, C_1 is a function of this coordinate. Let

$$a_1 = \mu_1 A_1, \qquad a_2 = \mu_2 A_2 \tag{1}$$

be the inner and the outer radii of the shell in the finitely deformed state caused by the static pressures q_1 and q_2 applied, respectively, on the inner and the outer curved surfaces. A material point at coordinates (r, θ, z) in the deformed state is at coordinates (R, θ, z) in the undeformed state. The corresponding stress field is given by (see, for example, the derivations in[13])

$$\begin{aligned} \tau^{11} &= -L(r) - q_1, \\ r^2\tau^{22} &= \Phi(Q^{-2} - Q^2) - L(r) - q_1, \\ \tau^{33} &= \Phi(1 - Q^2) - L(r) - q_1, \\ \tau^{12} &= \tau^{23} = \tau^{31} = 0, \end{aligned} \tag{2}$$

where

$$\Phi = 2\frac{\partial W}{\partial I}$$

$$L(r) = \int_{a_1}^{r} \Phi(Q^2 - Q^{-2})\frac{\mathrm{d}r}{r},$$

and, due to the incompressibility of the material,

$$Q = R/r = \frac{1}{r}[r^2 + A_1^2(1 - \mu_1^2)]^{1/2}. \tag{3}$$

The differential pressure $q = q_2 - q_1$ required to produce the prescribed deformation field is given by (Ref. [5], eqn (3)),

$$q = q_2 - q_1 = \int_{a_1}^{a_2} \Phi(Q^2 - Q^{-2})\frac{\mathrm{d}r}{r}. \tag{4}$$

We now consider a state of free, infinitesimal radial oscillations about the finitely deformed state such that the only nonzero incremental displacement component is $w_1 = w_1(r, t)$ in the radial direction. The induced incremental stress field is given by (see, for example, Ref. [13], Chpt. 4)

$$\begin{aligned} \hat{\tau}^{11} &= -2Pw_{1,r} + p', \\ r^2\hat{\tau}^{22} &= 2Pw_{1,r} + p', \\ \hat{\tau}^{33} &= p', \\ \hat{\tau}^{12} &= \hat{\tau}^{23} = \hat{\tau}^{31} = 0, \end{aligned} \tag{5}$$

where

$$P = -\Phi Q^2 - L(r) - q_1, \tag{6}$$

and p' is an unknown pressure.

The equations of motion in θ and z directions indicate that p' is a function of r and t only. Then, the secondary displacement field is governed by the equation of motion in the radial direction,

$$\Phi(Q^2 - Q^{-2})w_{1,rr} - 2\left[\Phi(Q^{-1} - Q)^2 - Q(1 - Q^2)\frac{\mathrm{d}\Phi}{\mathrm{d}Q}\right]\frac{1}{r}w_{1,r} + p'_{,r} = \rho\ddot{w}_1 \tag{7}$$

and by the incompressibility condition in the secondary deformation field,

$$w_{1,r} + \frac{1}{r}w_1 = 0. \tag{8}$$

In eqn (7), ρ denotes the current mass density, and a dot denotes differentiation with respect to time. Here, we note that eqn (7) reduces to eqn (5) of Ref. [5] if we substitute ϕ = const. in eqn (7) and v = zero in eqn (5) of Ref. [5] and make use of the equation $w_{1,rr} = -(2/r)w_{1,r}$ which is a direct result of the incompressibility condition.

The boundary conditions, obtained by the requirement that the secondary surface tractions vanish, are

$$2\Phi w_{1,r} + p' = 0 \quad \text{on} \quad r = a_1, a_2 \tag{9}$$

We seek solutions to the functions w_1 and p' in the form

$$\begin{aligned} w_1(r,t) &= u(r)\,e^{i\omega t}, \\ p'(r,t) &= f(r)\,e^{i\omega t} \end{aligned} \tag{10}$$

Satisfying eqns (7) and (8) by (10) results in two ordinary differential equations,

$$\Phi(Q^2 - Q^{-2})u_{,rr} - 2\left[\frac{\Phi(Q^{-1}-Q)^2}{r} - \frac{Q(1-Q^2)}{r}\frac{d\Phi}{dQ}\right]u_{,r} + f_{,r} = -\rho\omega^2 u, \quad u_{,r} + \frac{1}{r}u = 0, \tag{11}$$

which are solved for $u(r)$ and $f(r)$ in closed form:

$$\begin{aligned} u &= D/r, \\ f &= \left\{-\rho\omega^2 \ln r + \int \frac{2\Phi}{r^3}(1-Q^2)\left\{2 + \frac{Q}{\Phi}\frac{d\Phi}{dQ}\right\} dr\right\} D + E \end{aligned} \tag{12}$$

where D and E are integration constants. Substituting $u(r)$ and $f(r)$ into the boundary conditions, eqn (9), and requiring that the determinant of the coefficient matrix vanishes for a non-trivial solution, we obtain the characteristic equation of the problem, and hence an expression for the frequencies of small oscillations;

$$\omega^2 = \frac{8}{\rho A_1^2 \ln\left[\frac{(1-\lambda^2\bar{K})}{\lambda^2(1-\bar{K})}\right]} \int_{Q_2}^{Q_1} \Phi Q^3\, dQ \tag{13}$$

where $\bar{K} = 1 - \mu_1^2$, $\lambda = A_1/A_2$. If the material is homogeneous, then Φ is constant and the resulting expression corresponds to eqn (29) of Ref. [5]. Furthermore, if the primary deformation is infinitesimal, then $\bar{K} \to 0$, and the expression thus obtained reduces to the well known classical result of small radial oscillations of cylindrical shells (Ref. [9], eqn (37)).

For simplicity, we now assume a quadratic form for Φ. If Φ_i, Φ_m, and Φ_0 denote, respectively, the material constants on the inner, middle, and the outer surfaces, then we can write

$$\Phi = \Phi_i\left\{B_1\left(\frac{R}{A_1}\right)^2 + B_2\left(\frac{R}{A_1}\right) + B_3\right\}, \tag{14}$$

where

$$B_1 = \frac{\lambda^2}{(1-\lambda)^2}\{2(\Phi_0/\Phi_i) - 4(\Phi_m/\Phi_i) + 2\}, \tag{15}$$

$$B_2 = \frac{\lambda}{1-\lambda}(\Phi_0/\Phi_i - 1) - \frac{1+\lambda}{\lambda}B_1, \tag{16}$$

$$B_3 = 1 - B_1 - B_2. \tag{17}$$

Substituting eqn (14) into eqn (13) and noting that

$$\begin{aligned} R/A_1 &= \bar{K}^{1/2}Q/(Q^2-1)^{1/2}, \\ Q_1 &= A_1/a_1 = 1/(1-\bar{K})^{1/2}, \\ Q_2 &= A_2/a_2 = 1/(1-\lambda^2\bar{K})^{1/2}, \end{aligned} \tag{18}$$

we obtain

$$\omega^2 = \frac{8\Phi_i}{\rho A_1^2 \ln\left\{\frac{(1-\lambda^2\bar{K})}{\lambda^2(1-\bar{K})}\right\}} \left\{B_1\bar{K}\int_{Q_2}^{Q_1} \frac{Q^5}{(Q^2-1)}\,dQ + B_2\bar{K}^{1/2}\int_{Q_2}^{Q_1} \frac{Q^4}{(Q^2-1)^{1/2}}\,dQ + B_3\int_{Q_2}^{Q_1} Q^3\,dQ\right\}. \tag{19}$$

Performing the integration, for $\bar{K} \neq 0$, the non-dimensionalized frequency expression is given by

$$\begin{aligned}\bar{\omega}^2 = \frac{2}{\ln\left\{\frac{(1-\lambda^2\bar{K})}{\lambda^2(1-\bar{K})}\right\}} & \left\{B_1\left\{\frac{3-2\bar{K}}{(1-\bar{K})^2} - \frac{3-2\lambda^2\bar{K}}{(1-\lambda^2\bar{K})^2} + 2\ln\frac{(1-\lambda^2\bar{K})}{\lambda^2(1-\bar{K})}\right\}\right. \\ & + B_2\left\{\frac{5-3\bar{K}}{2(1-\bar{K})^2} - \frac{5\lambda-3\lambda^3\bar{K}}{2(1-\lambda^2\bar{K})^2} + \frac{2}{2\sqrt{\bar{K}}}\ln\frac{(1+\sqrt{\bar{K}})\sqrt{(1-\lambda^2K)}}{(1+\lambda^2\sqrt{\bar{K}})\sqrt{(1-\bar{K})}}\right\} \\ & \left. + B_3\left\{\frac{\lambda^4\bar{K}-2\lambda^2+2-\bar{K}}{(1-\bar{K})^2(1-\lambda^2\bar{K})^2}\right\}\right\},\end{aligned} \tag{20}$$

and for $\bar{K} = 0$,

$$\bar{\omega}^2 = \frac{2}{\ln\frac{1}{\lambda^2}}\left\{2B_1\ln\frac{1}{\lambda^2} + B_2\left(4 - \frac{5}{2}\lambda - \frac{3}{2}\lambda^2\right) + 2B_3(1-\lambda^2)\right\}, \tag{21}$$

where

$$\bar{\omega}^2 = \rho A_1^2\omega^2/\Phi_i. \tag{22}$$

2. *Spherical shells*†

In this section we consider a spherical shell whose undeformed inner and outer radii are denoted, respectively, by A_1 and A_2. In the spherically symmetric deformation state produced by an internal pressure q_1 and an external pressure q_2, the new inner and outer radii are $a_1 = \mu_1 A_1$, $a_2 = \mu_2 A_2$. For a neo-Hookean material, we have

$$q = q_2 - q_1 = 2\int_{Q_2}^{Q_1} \Phi(1+Q^3)\,dQ \tag{23}$$

The secondary dynamic displacement field $w_1 = w_1(r,t)$, $w_2 = w_3 = 0$ is now superposed onto the initially deformed shell. The equations governing this secondary state are

$$\Phi Q^4 w_{1,rr} + \left\{\frac{\Phi}{r}(8Q - 4Q^4 - Q^{-2}) - \frac{2Q^2(Q^3-1)}{r}\frac{d\Phi}{dQ}\right\} w_{1,r} + p'_{,r} = \rho\ddot{w}_1, \tag{24}$$

and the incompressibility condition

$$w_{1,r} + \frac{2}{r}w_1 = 0, \tag{25}$$

where

$$\begin{aligned} p' &= p'(r,t), \\ Q = R/r &= \frac{1}{r}[r^3 + A_1^3(1-\mu_1^3)]^{1/3}. \end{aligned} \tag{26}$$

As in the case of cylindrical shells, eqn (24) reduces to eqn (15) of Ref. [5] if we substitute Φ = const. in eqn (24), assume spherical symmetry and make use of the incompressibility condition in the latter. In this case, eqn (16) of Ref. [12] also becomes identical with eqn (24).

†The details of derivations are omitted here since they are essentially the same as those given in Ref. {12}.

However, the term introducing the material nonhomogeneity (i.e. the term including $d\Phi/dQ$) is erroneous in eqn (16) of Ref. [12].

The boundary conditions governing the secondary displacement field are given by

$$2\Phi Q^4 w_{1,r} + p' = 0 \quad \text{on} \quad r = a_1, a_2. \tag{27}$$

Assuming again $w_1 = u(r)\,e^{i\omega t}$ and $p' = f(r)\,e^{i\omega t}$, and solving eqns (24) and (25) subjected to the boundary conditions, eqn (27), we obtain

$$\omega^2 = \frac{2(1-\bar{K})^{1/3}(1-\lambda^3\bar{K})^{1/3}}{\rho A_1^2\{(1-\lambda^3\bar{K}) - \lambda(1-\bar{K})^{1/3}\}}\cdot\frac{1}{\bar{K}}\int_{Q_2}^{Q_1}\Phi(7Q^6-1)\,dQ \tag{28}$$

where $\bar{K} = 1-(a_1/A_1)^3$.

In order to compare results with those given in [12], we now choose a cubic variation for the material constant in the following form

$$\Phi = B_0 + B_3\left(\frac{R}{A_1}\right)^3. \tag{29}$$

The corresponding frequency expression for $\bar{K} \neq 0$ is given by

$$\begin{aligned}\bar{\omega}^2 = \frac{\rho A_1^2\omega^2}{B_0} = \frac{2(1-\bar{K})^{1/3}(1-\lambda^3\bar{K})^{1/3}}{(1-\lambda^3\bar{K})^{1/3}-\lambda(1-\bar{K})^{1/3}}\Bigg\{&\frac{(2-\bar{K})}{(1-\bar{K})^{7/3}} - \frac{\lambda^3(2-\lambda^3\bar{K})}{(1-\lambda^3\bar{K})^{7/3}}\\ &+\frac{B_3}{B_0}\Bigg[\frac{24(1-\bar{K})^2+7(1-\bar{K})+4}{4(1-\bar{K})^{7/3}} - \frac{24(1-\lambda^3\bar{K})^2+7(1-\lambda^3\bar{K})+4}{4(1-\lambda^3\bar{K})^{7/3}}\\ &+3\ln\frac{\lambda\{1-(1-\bar{K})^{1/3}\}}{\{1-(1-\lambda^3\bar{K})^{1/3}\}} + 2\sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}(1-\lambda^3\bar{K})^{7/3}}+\frac{1}{\sqrt{3}}\right)\\ &-2\sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}(1-\bar{K})^{1/3}}+\frac{1}{\sqrt{3}}\right)\Bigg]\Bigg\}.\end{aligned} \tag{30}$$

When $\bar{K} = 0$, eqn (30) becomes indeterminate. However, using L'Hospital rule, for $\bar{K} = 0$ we obtain

$$\bar{\omega}^2 = \frac{4}{(1-\lambda)}\left\{(1-\lambda^3) + 3\frac{B_3}{B_0}\ln\frac{1}{\lambda}\right\}. \tag{31}$$

If the material is homogeneous, then $B_3 = 0$ and eqn (31) reduces to the classical result of pure infinitesimal radial oscillations for spherical shells (Ref. [14], eqn (3.2)).

DISCUSSION OF THE RESULTS

1. *Cylindrical shells*

Figure 1 shows $\bar{\omega}^2$ as plotted against the deformation parameter $\bar{K} = 1-\mu_1^2$ for various A_1/A_2 ratios corresponding to $\Phi = \Phi_i\{B_1(R/A_1)^2\}$ (i.e. when only the quadratic term is retained). It is seen that, for a given A_1/A_2 ratio $\bar{\omega}^2$ increases sharply with an increasing net inward pressure ($0 < \bar{K} < 1$) while it decreases with an increasing net outward pressure ($-\infty < \bar{K} < 0$). In the latter, the frequency approaches zero asymptotically as $\bar{K}$ approaches $-\infty$. This indicates that the shell softens when inflated and it fails without bound in a mode of radial expansion. An interesting observation is that when $\bar{K} = 0$, $\bar{\omega}^2 = \rho A_1^2\omega^2/\Phi_i = 4$ for all A_1/A_2 ratios.

The results corresponding to a nonhomogeneity function given by $\Phi = \Phi_i\{B_1(R/A_1)^2 + B_3\}$ are shown in Fig. 2 for a shell of $A_1/A_2 = 0.80$. The shell shows a hardening behaviour in the compression region and a softening behaviour in the tension region for all ϕ_0/ϕ_i ratios. Noting that $\Phi_0/\Phi_i = 1$ corresponds to the homogeneous case, we observe that compared to the homogeneous case, the frequencies are higher when $\Phi_0 > \Phi_i$ and lower when $\Phi_0 < \Phi_i$.

Figure 3 shows $\bar{\omega}^2$ as plotted against $\bar{K}$ for $A_1/A_2 = 0.80$ with a nonhomogeneity function $\Phi = \Phi_i\{B_2(R/A_1) + B_3\}$. For $0 < \bar{K} < 1$, i.e. in the compression region, the shells show a definite

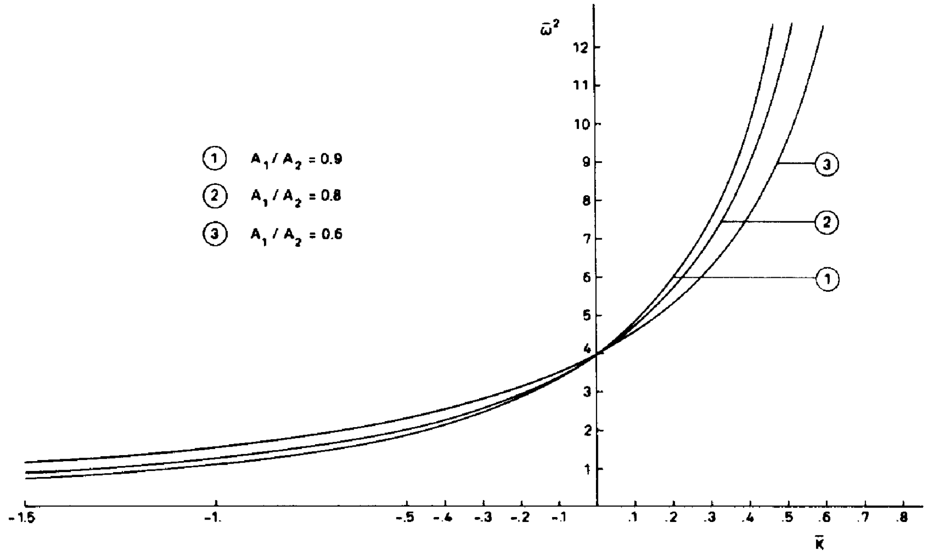

Fig. 1. Frequency versus initial deformation parameter for a cylindrical shell of various wall thicknesses (only the quadratic term is retained).

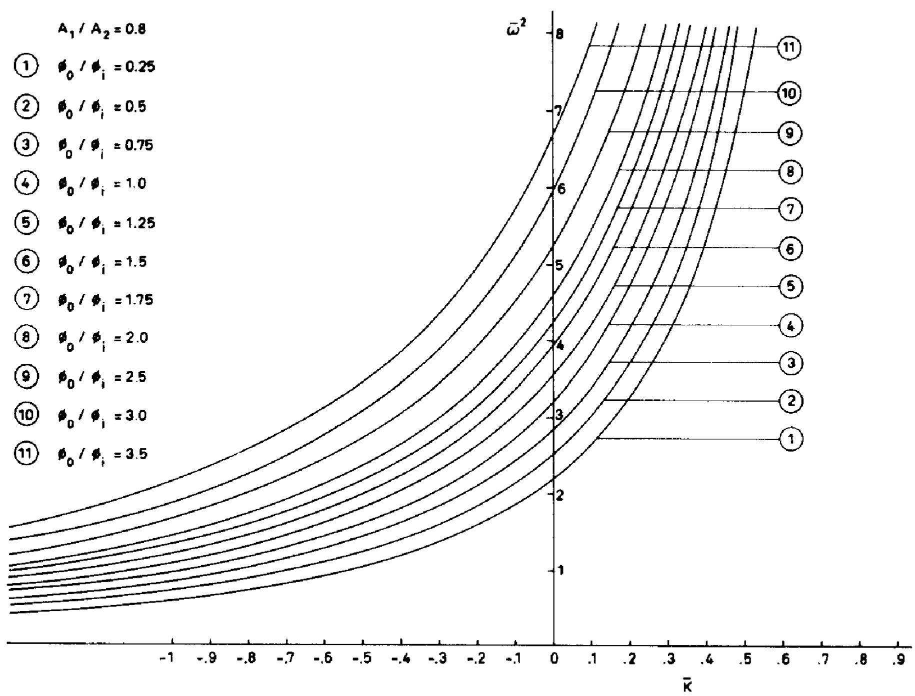

Fig. 2. Frequency versus initial deformation parameter of a cylindrical shell, for various nonhomogeneity factors (no linear term).

hardening behavior when $\phi_0/\phi_i \leq 1$, $\Phi_0/\Phi_i = 1$ corresponding to the homogeneous case. For $\Phi_0/\Phi_i > 1$, however, the shells show a softening behavior in the early compression regions, and a hardening behavior as $\bar{K}$ becomes larger. A close examination of eqn (30) shows that $\bar{\omega}^2$ becomes a complex number for $\bar{K} < 0$ due to the presence of $\bar{K}^{1/2}$ associated with the linear term. For reasons which we are unable to explain physically, this forces us mathematically to conclude that pure radial oscillations cannot exist for net outward pressure, $\bar{K} < 0$, when there is a linear variation of the material constant as a function of the radial coordinate.

2. *Spherical shells*

In order to compare the results with those of Ref. [12], a representative example is worked out for $A_1/A_2 = 0.80$ for various nonhomogeneity distributions, $B_3/B_0 = 0$ (homogeneous case), 0.25,

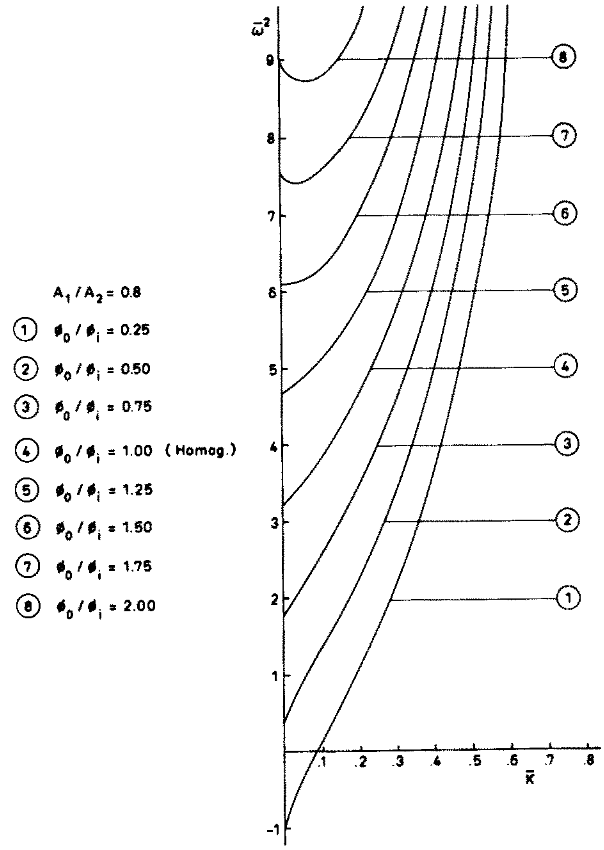

Fig. 3. Frequency versus initial deformation parameter for a cylindrical shell, for various nonhomogeneity factors (general linear variation).

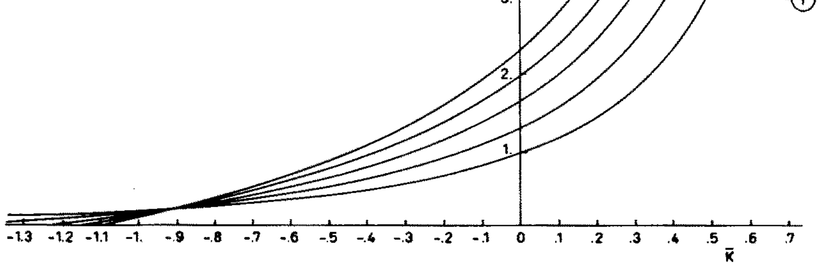

Fig. 4. Frequency versus initial deformation parameter of a spherical shell, for various nonhomogeneity factors ($\phi = B_0 + B_3(R/A_1)^3$).

0.50, 0.75 and 1.0. In all cases the shell hardens in the compression region and softens in the tension region, and it fails $\bar{\omega}^2 = 0$) in a mode of radial expansion. The curves do not show the erratic behavior indicated in[12].

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